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A combined viscoelastic–viscoplastic behavior of particle reinforced composites

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ABSTRACT

This study formulates a micromechanical model for predicting effective viscoelastic–viscoplastic responses of composites. The studied composites consist of solid spherical particle reinforcements dispersed in a homogeneous matrix. The particle constituent is assumed linear elastic, while the matrix exhibits combined viscoelastic–viscoplastic responses. The Schapery integral model is used for the 3D isotropic non-linear viscoelastic responses. Two viscoplastic models are considered: the Perzyna model, having a rate-independent yield surface and an overstress function, and the Valanis endochronic model based on an irreversible thermodynamics without a yield surface. The Valanis model is suitable for materials when viscoplastic responses occur at early loadings (small stress levels). A unit-cell model with four particle and polymer sub-cells is generated to obtain homogenized responses of the particle-reinforced composites. Available micromechanical models and experimental data in the literature are used to verify the proposed micromechanical model in predicting effective time-dependent and inelastic responses of composites. Field variables in the homogenized composites are compared to the ones in heterogeneous composites. The heterogeneous composites, having detailed particle geometries, are modeled using finite element (FE) method.

1. Introduction

Polymer and metal matrix composites have been utilized in many engineering applications that involve various temperature ranges. An important issue concerning such composites is their degradation at elevated temperatures. Under relatively high stress levels and elevated temperatures, polymer and metal matrix composites exhibit time-dependent and inelastic deformations. Unlike in monolithic structural materials, such as unreinforced metal and polymer, composites form heterogeneous systems with distinct microstructural geometries and constituent properties. The heterogeneity in composites, driven by the different properties of the constituents and various microstructural arrangements, makes predicting effective time-dependent and inelastic behaviors of composites more complicated. We study a combined viscoelastic–viscoplastic behavior of particle-reinforced composites. The viscoelastic strain component consists of a recoverable- and thermodynamically reversible part (elastic strain) and a recoverable- and dissipative deformation part, as discussed in Mareau et al. (2009). When an inelastic strain, which is unrecoverable and thermodynamically irreversible, is assumed to depend only on the magnitude of the local stress or strain, the term 'plastic strain' is used. When the plastic deformation also changes with time, like in the viscous component of the viscoelastic part, we use the term 'viscoplastic strain.'

Micromechanical models have been widely used to determine effective mechanical responses of composites by taking into account detailed microstructural geometries and constitutive models of the constituents. Micromechanical formulations for predicting elastic–plastic or viscoplastic responses of particle reinforced composites have been derived for metal–matrix composite. Some of these micromechanical models have been extended to predict inelastic or viscoplastic behaviors of polymer based composites. Weng (1993) used the self-consistent method for analyzing effective creep behavior of composites. A spherical inclusion is embedded in polycrystal matrix. Inclusion and matrix exhibit different linear viscoelastic behavior. It was assumed that the inclusion creeps less than the matrix creeps. Ju and Chen (1994) presented a micromechanical framework to predict effective elasto-plastic behavior of two-phase particle reinforced metal matrix composites. An overstress concept was used for the plastic response. Particle was assumed to be linear elastic and matrix exhibited an elasto-plastic behavior. To obtain effective responses, the Eigenstrain concept of Eshelby (1957), ensemble volume average concept, and closed form micromechanical equation with interaction between particle and matrix were used. Ju and Tseng (1996) developed a micromechanical model of effective inelastic material properties that take into account inter particle interaction based on an ensemble volume averaging method. The Von-Misses plasticity constitutive model was used for the elasto-plastic response in

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the matrix constituent. When plastic deformations are considered, a stress–strain at each material point in the matrix depends on its location and history of deformation. The existence of neighboring particles could induce non-uniformly distributed stresses in the matrix medium due to the inter-particle reactions. In predicting overall responses of the heterogeneous materials, Ju and Tseng (1996) statistically incorporated the effect of these inter-particle reactions on the local strain of each material point in the matrix medium through the formulation of the fourth order tensor. This tensor relates the effective strain to the ensemble-volume averaged Eigen strain. The micromechanical model predictions were verified using experimental data for composites with several particle volume contents. Ghosh and Mukhopadhyay (1991) have proposed Voronoi Cell Finite Element Method (VCFEM) to predict effective properties of heterogeneous materials. A representative microstructure of materials was divided into a network of multi sided convex elements called "Voronoi polygons or Cells". These Voronoi polygons are treated as elements in the FE scheme. To model multi-phase materials, heterogeneities or inclusions can be included to each Voronoi polygon. The Voronoi element has been coupled to asymptotic homogenization scheme for multi-scale analyses of heterogeneous materials. Ghosh et al. (1995) have extended the VCFEM for analyzing elastic–plastic behavior of heterogeneous materials.

Seelig and Van der Giessen (2002) investigated elasto-viscoplastic responses of acrylonitrile–butadiene–styrene (ABS) particles embedded in a polycarbonate (PP). The ABS rubber particles were considered as voids because of their low modulus. PP matrix exhibits elasto-viscoplastic behavior. The equivalent elasto-viscoplastic responses of ABS are obtained via the Mori–Tanaka model. The homogenized elasto-viscoplastic responses are compared to the ones obtained from the representative volume element (RVE) models of the composites. The RVE models were generated using 2D FE. Danielsson et al. (2007) used the Voronoi cell model to predict elasto-viscoplastic responses of rubber toughened glassy polymers. Periodic boundary condition is imposed to the representative microstructure of the composites. The viscoplastic flow of the porous glassy polymer is characterized by a plastic strain rate potential, which followed a power-law model of Hutchinson (1976). Pierard et al. (2007) developed a linearized homogenization method to predict elasto-viscoplastic response of particle reinforced composites. A total strain for a small deformation gradient problem was decomposed into elastic and viscoplastic components. The Laplace–Carson transformation was used to obtain time dependent response from the thermo-elasticity solution. The Mori–Tanaka homogenization method was applied to provide the effective properties. The effective responses from the homogenization procedure are compared with the ones determined from detailed FE microstructures. Mareau et al. (2009) presented an incremental formulation for solving the governing equations of non-linear material behaviors, i.e., elastic, elastoplastic, and viscoplastic behaviors. The constitutive models of the constituents were represented by spring-dashpot mechanical analog models. They extended the coupled translation field technique and self-consistent approximation method, proposed by Sabar et al. (2002) for elastic–viscoplastic behaviors, to predict overall viscoelastic–viscoplastic responses of heterogeneous materials. Responses obtained from this method are compared to the ones available in the literature for linear behaviors.

Some polymers used as constituents in composite systems exhibit combined viscoelastic–viscoplastic responses, e.g. high density polyethylene (Lai and Bakker, 1995) and polycarbonate (Frank, 1998). These combined responses can occur at early loading (small stress/strain levels). Depending on the materials being studied, there are thresholds that initiate the viscoplastic deformation. For some polymers, loaded for a sufficient period, viscoplastic deformation can occur, in addition to the viscoelastic deformation, even under low applied stresses. This is due to the macro-molecular arrangements of the polymers. When materials exhibit a combined viscoelastic and viscoplastic response, the plastic deformation also increases with time during loading and upon removal of the load only the viscoelastic strain is recovered. In order to examine the viscoelastic and viscoplastic deformations it is then necessary to perform loading–unloading tests. Characterization of viscoelastic and viscoplastic material properties from creep-recovery tests at several stress levels can be found in Lai and Bakker (1995). Micromechanical models for a combined viscoelastic–viscoplastic response are very limited. Aboudi (2005) has developed a micromechanical model to predict the viscoelastic–viscoplastic responses of multiphase materials. The viscoelastic–viscoplastic model for polymers developed by Frank and Brockman (2001) is implemented in the multiphase composites. An asymptotic expansion homogenization (AEH) method is used to obtain effective properties. Depending on the complexity of the composite's microstructure, the AEH method could results in high computational cost. While micromechanical models for predicting elastic–plastic and viscoplastic behaviors have been extensively developed for polycrystal materials and metal matrix composites, limited studies have been done on understanding time-dependent and inelastic responses of polymer-based composites.

This study introduces a simplified micromechanical formulation for predicting a combined viscoelastic–viscoplastic response of particle reinforced composites. The studied composites consist of linear elastic solid spherical particles and viscoelastic–viscoplastic matrix constituent. A previously developed micromechanical model of viscoelastic particle composites (Muliana and Kim, 2007) is extended for the combined viscoelastic–viscoplastic response of composites. The new contribution of this manuscript is in integrating the time-dependent and inelastic deformation of the matrix constituent to the simplified micromechanical model for particle composites. When perfect bonding between the inclusion and matrix is imposed, different properties in the constituents generate internal stresses. The existence of the internal stresses in the matrix constituent causes not only viscoelastic deformation but also time-dependent plastic deformation, affecting overall performance of the composites. Furthermore, when the prescribed loadings are removed, the existence of the plastic deformation remains in the matrix constituent, which in turn induces residual stresses. The aim of the proposed simplified micromechanical formulation is to be able to predict overall time-dependent and inelastic responses of particle-reinforced composites and at the same time incorporate histories of deformation at both constituents. Experimental data and analytical solutions available in the literature are used to verify the micromechanical model formulation. The responses obtained from the present micromechanical model represent behaviors of a fictitiously homogenized composite. These responses are compared to the ones of heterogeneous composites having detailed particle micro-structural models. The heterogeneous composite models are generated using FE. This manuscript is organized as follows. Section 2 presents constitutive material models for the combined viscoelastic–viscoplastic responses for the matrix constituents. A time-integration algorithm is formulated to obtain solutions for the field variables. Section 3 discusses the micromechanical model formulation. Verification of the time-integration algorithm and micromechanical model in predicting effective time-dependent and inelastic responses of composites is given in Section 4. Field variables, i.e., stress and strain, of the homogenized composites are also compared to the ones of the heterogeneous composites. Conclusion and discussion are presented in Section 5.

2. Constitutive model for viscoelastic–viscoplastic isotropic material

This section introduces a time integration algorithm for solving a combined viscoelastic–viscoplastic constitutive equation of isotropic constituents. The algorithm is derived based on an implicit time integration method within a general displacement based FE analysis and suitable for small strain problems. When materials exhibit a combined viscoelastic and viscoplastic response, plastic deformation in addition to the viscoelastic deformation increases with time during loading and upon unloading only the viscoelastic strain is recovered. The Schapery integral model is used for the 3D isotropic non-linear viscoelastic responses. Two viscoplastic models are considered: the Perzyna model, having a rate-independent yield surface and an overstress function, and the Valanis endochronic model based on an irreversible thermodynamics without a yield surface. The Valanis model is suitable for materials when viscoplastic responses occur at early loadings. Linearized solutions of the non-linear constitutive equations and iterative schemes are formulated. The linearized relation is used as a starting point to calculate trial stress–strain solutions in every time increment. The purpose of forming iterative schemes is to minimize errors arising from the linearization.

In small strain theory, the total strains and incremental strains at a material point can be additively decomposed into viscoelastic and viscoplastic components:

$$\begin{aligned} \varepsilon_{ij}^{t} &= \varepsilon_{ij}^{ve,t} + \varepsilon_{ij}^{vp,t} \\ \Delta\varepsilon_{ij}^{t} &= \Delta\varepsilon_{ij}^{ve,t} + \Delta\varepsilon_{ij}^{vp,t} \quad \forall t \geqslant 0 \\ \varepsilon_{ij}^{t} &= \varepsilon_{ij}^{t-\Delta t} + \Delta\varepsilon_{ij}^{t} = \varepsilon_{ij}^{ve,t-\Delta t} + \Delta\varepsilon_{ij}^{ve,t} + \varepsilon_{ij}^{vp,t-\Delta t} + \Delta\varepsilon_{ij}^{vp,t} \end{aligned} \tag{1}$$

where $\varepsilon_{ij}^{ve,t}$ and $\varepsilon_{ij}^{vp,t}$ are the viscoelastic and viscoplastic strains at current time t, respectively; $\Delta\varepsilon_{ij}^{ve,t}$ and $\Delta\varepsilon_{ij}^{vp,t}$ are the current incremental viscoelastic and viscoplastic strains. The rest of this manuscript will denote time-dependence of variables with superscript of the time variable.

2.1. Schapery viscoelastic model

A single-integral constitutive equation derived from the thermodynamics of irreversible process, Schapery (1969), is used for the viscoelastic component. This integral model is applicable for small strain problems and allows incorporating stress or strain dependent viscoelastic responses. The stress or strain dependent material parameters can be determined from a set of creep or relaxation data. The Schapery non-linear single integral equation is generalized for a non-linear isotropic viscoelastic response and is written as:

$$\varepsilon_{ij}^{ve,t} = e_{ij}^{ve,t} + \frac{1}{3}\delta_{ij}\varepsilon_{kk}^{ve,t} \tag{2}$$

$$e_{ij}^{ve,t} = \frac{1}{2}g_0(\bar{\sigma}^t)J_0 S_{ij}^t + \frac{1}{2}g_1(\bar{\sigma}^t)\int_0^t \Delta J^{(\psi^t-\psi^\tau)}\frac{d[g_2(\bar{\sigma}^\tau)S_{ij}^\tau]}{d\tau}d\tau \tag{3}$$

$$\varepsilon_{kk}^{ve,t} = \frac{1}{3}g_0(\bar{\sigma}^t)B_0\sigma_{kk}^t + \frac{1}{3}g_1(\bar{\sigma}^t)\int_0^t \Delta B^{(\psi^t-\psi^\tau)}\frac{d[g_2(\bar{\sigma}^\tau)\sigma_{kk}^\tau]}{d\tau}d\tau \tag{4}$$

It is assumed that loading starts at time $t = 0$ and the material is non-aging. The parameters J_o and B_o are the instantaneous elastic shear and bulk compliances, respectively. The terms ΔJ and ΔB are the time-dependent (transient) shear and bulk compliances, respectively. The non-linear parameters g_0, g_1, and g_2 of the multi-axial behaviors are modeled as a function of the current effective stress $\bar{\sigma}^t$. The corresponding linear elastic Poisson's ratio, v, obtained from the creep test is assumed to be time independent. The shear and bulk compliances are expressed as:

$$\begin{aligned} J_0 &= 2(1+\upsilon)D_0, \quad B_0 = 3(1-2\upsilon)D_0 \\ \Delta J^{\psi^t} &= 2(1+\upsilon)\Delta D^{\psi^t}, \quad \Delta B^{\psi^t} = 3(1-2\upsilon)\Delta D^{\psi^t} \end{aligned} \tag{5}$$

A Prony series of exponential functions is used for the transient part due to the advantage of this representation in solving the integral form in Eqs. (3) and (4) in a recursive manner. The uniaxial transient compliance is expressed as:

$$\Delta D^{\psi^t} = \sum_{n=1}^{N} D_n(1-\exp[-\lambda_n\psi^t]) \tag{6}$$

where ψ is the reduced-time (effective time), given by:

$$\psi^t \equiv \psi(t) = \int_0^t \frac{ds}{a_\sigma^{\sigma^s}} \tag{7}$$

The parameter a_σ^t is the time-stress shift factor. It is also assumed that all material parameters and field variables before time $t = 0$ are equal to zero. A recursive-iterative method is used for solving the non-linear viscoelastic model in Eqs. (3) and (4). Detailed recursive-iterative algorithm for the non-linear viscoelastic behaviors is presented in Haj-Ali and Muliana (2004). The incremental viscoelastic strain derived from the recursive-iterative approach is summarized as:

$$\begin{aligned} \Delta\varepsilon_{ij}^{ve,t} = &\left[\bar{J}^t\sigma_{ij}^t + \frac{1}{3}\delta_{ij}\{\bar{B}^t - \bar{J}^t\}\sigma_{kk}^t\right] \\ &- \left[\bar{J}^{t-\Delta t}\sigma_{ij}^{t-\Delta t} + \frac{1}{3}\delta_{ij}\{\bar{B}^{t-\Delta t} - \bar{J}^{t-\Delta t}\}\sigma_{kk}^{t-\Delta t}\right] - A_{ij}^t - \frac{1}{3}B^t\delta_{ij} \end{aligned} \tag{8}$$

$$\bar{J}^t = \frac{1}{2}\left[g_0^t J_0 + g_1^t g_2^t\sum_{n=1}^{N}J_n - g_1^t g_2^t\sum_{n=1}^{N}J_n\frac{1-\exp[-\lambda_n\Delta\psi^t]}{\lambda_n\Delta\psi^t}\right] \tag{9}$$

$$\bar{B}^t = \frac{1}{3}\left[g_0^t B_0 + g_1^t g_2^t\sum_{n=1}^{N}B_n - g_1^t g_2^t\sum_{n=1}^{N}B_n\frac{1-\exp[-\lambda_n\Delta\psi^t]}{\lambda_n\Delta\psi^t}\right] \tag{10}$$

$$\begin{aligned} A_{ij}^t = &\frac{1}{2}\sum_{n-1}^{N}J_n\left(g_1^t\exp[-\lambda_n\Delta\psi^{t-\Delta t}] - g_1^{t-\Delta t}\right)q_{ij,n}^{t-\Delta t} + \frac{1}{2}g_2^{t-\Delta t}\sum_{n=1}^{N}J_n \\ &\times\left[g_1^{t-\Delta t}\left(\frac{1-\exp[-\lambda_n\Delta\psi^{t-\Delta t}]}{\lambda_n\Delta\psi^{t-\Delta t}}\right) - g_1^t\left(\frac{1-\exp[-\lambda_n\Delta\psi^t]}{\lambda_n\Delta\psi^t}\right)\right]S_{ij}^{t-\Delta t}] \end{aligned} \tag{11}$$

$$\begin{aligned} B^t = &\frac{1}{2}\sum_{n-1}^{N}B_n\left(g_1^t\exp[-\lambda_n\Delta\psi^{t-\Delta t}] - g_1^{t-\Delta t}\right)q_{kk,n}^{t-\Delta t} + \frac{1}{2}g_2^{t-\Delta t}\sum_{n=1}^{N}B_n \\ &\times\left[g_1^{t-\Delta t}\left(\frac{1-\exp[-\lambda_n\Delta\psi^{t-\Delta t}]}{\lambda_n\Delta\psi^{t-\Delta t}}\right) - g_1^t\left(\frac{1-\exp[-\lambda_n\Delta\psi^t]}{\lambda_n\Delta\psi^t}\right)\right]\sigma_{kk}^{t-\Delta t} \end{aligned} \tag{12}$$

Material parameters $\bar{J}^t$ and $\bar{B}^t$ are the shear and bulk compliances that depend on the effective stress at the current time t, while A_{ij}^t and B^t are the strain components that contain history variables. The incremental reduced time is expressed by $\Delta\psi^t \equiv \psi^t - \psi^{t-\Delta t}$. Variables $q_{ij,n}^{t-\Delta t}$ and $q_{kk,n}^{t-\Delta t}$ are the shear and volumetric hereditary variables for every Prony series term at the end of the previous time, $t - \Delta t$. At the end of each time interval, the hereditary variables are updated and stored, which will be used for the next time integration step. The updated hereditary variables are:

$$\begin{aligned} q_{ij,n}^t = &\exp[-\lambda_n\Delta\psi^t]q_{ij,n}^{t-\Delta t} \\ &+ \left(g_2^t S_{ij}^t - g_2^{t-\Delta t}S_{ij}^{t-\Delta t}\right)\frac{1-\exp[-\lambda_{ij,n}\Delta\psi^t]}{\lambda_{ij,n}\Delta\psi^t} \end{aligned} \tag{13}$$

$$\begin{aligned} q_{kk,n}^t = &\exp[-\lambda_n\Delta\psi^t]q_{kk,n}^{t-\Delta t} \\ &+ \left(g_2^t\sigma_{kk}^t - g_2^{t-\Delta t}\sigma_{kk}^{t-\Delta t}\right)\frac{1-\exp[-\lambda_{ij,n}\Delta\psi^t]}{\lambda_{ij,n}\Delta\psi^t} \end{aligned} \tag{14}$$

2.2. Perzyna viscoplastic model

Many structural materials are often exposed to long-term mechanical loading and elevated temperatures. Under such loading and environmental conditions, materials could experience significant time-dependent and inelastic deformations. Several viscoplastic constitutive models, based on an overstress assumption, have been formulated to describe time-dependent and inelastic behaviors of metallic materials. Perzyna's viscoplastic model has been used to predict response of metals under a wide range of strain rates and temperature changes. The Perzyna viscoplastic model is shown to be applicable to simulate time-dependent and inelastic responses of some polymer (Chailleux and Davies, 2003, 2005). The viscoplastic strain rate in the Perzyna (1966, 1971) model for isotropic material is expressed as:

$$\dot{\varepsilon}_{ij}^{vp,t} = \dot{\lambda}^t \frac{\partial F(\bar{\sigma}^t, k^t)}{\partial \sigma_{ij}^t} \tag{15}$$

where $\dot{\varepsilon}_{ij}^{vp,t}$ is the viscoplastic strain rate at current time t and $\dot{\lambda}^t$ is the magnitude of the viscoplastic strain rate, also known as a plastic multiplier. A yield function $F(\bar{\sigma}^t, k^t)$ is expressed in terms of an effective stress:

$$F(\bar{\sigma}^t, k^t) = \bar{\sigma}^t - \sigma_y^o - hk^t \tag{16}$$

The parameter σ_y^o is the initial yield stress measured from a uniaxial loading and h is the hardening material constant. The hardening material parameter can also depend on the current effective stress $\bar{\sigma}^t$. The parameter k^t is the accumulated effective viscoplastic strain, expressed as:

$$\begin{aligned} k^t &= \textstyle\int_0^t \dot{k}^s \, ds \\ \dot{k}^t &= \dot{e}^{vp,t} = \sqrt{\tfrac{2}{3} \dot{e}_{ij}^{vp,t} \dot{e}_{ij}^{vp,t}} \end{aligned} \tag{17}$$

Next, the plastic multiplier is obtained from the following equation:

$$\dot{\lambda} = \frac{1}{\eta_p} \langle \Phi(F) \rangle e^{-\frac{h}{\eta_p} t} \tag{18}$$

where η_p is the viscosity coefficient during the viscoplastic deformation, < > represents the Macauley bracket, the function $\Phi(F)$ depends on the distance of the current stress point to the yield surface (Bathe, 1996; Simo et al., 1998). Perzyna (1966) proposed several functions for $\Phi(F)$, e.g., linear, power law, and exponential forms. The function can be characterized by fitting available experimental data. For example, Kojic and Bathe (2005) have shown that a power law model can be used to fit viscoplastic strains based on creep tests. Following Kojic and Bathe (2005), for an isotropic hardening viscoplastic model the function $\Phi(F)$ is defined by:

$$\Phi(F) = \left[\frac{\bar{\sigma}^t - \sigma_y^o - hk^t}{\sigma_y^o} \right]^n \tag{19}$$

where n, h, and σ_y^o are material constants, which need to be calibrated by fitting experimental data. Kim and Muliana (2009) have formulated time-integration algorithm for the Perzyna viscoplastic model, which is compatible with a displacement based FE framework. The incremental viscoplastic strain for the Perzyna is given as:

$$\Delta\varepsilon_{ij}^{vp,t} = \frac{\Delta t}{\eta_p} \langle \Phi(F) \rangle \frac{3}{2\bar{\sigma}^t} \left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl} \right) S_{kl}^t e^{-\frac{h}{\eta_p} t} \tag{20}$$

The current stress is determined from stress at previous time step and incremental stress at current time, $(\sigma_{ij}^t = \sigma_{ij}^{t-\Delta t} + \Delta\sigma_{ij}^t)$, and plastic strain is updated similarly. The current $\Phi(F)$ in an incremental formulation is then defined as:

$$\Phi(F) = \left[\frac{(\bar{\sigma}^{t-\Delta t} + \Delta\bar{\sigma}^t) - \sigma_y^o - h(k^{t-\Delta t} + \Delta k^t)}{\sigma_y^o} \right]^n = [\varphi]^n \tag{21}$$

Substituting Eqs. (8) and (20) into Eq. (1), the total incremental strains $\Delta\varepsilon_{ij}^t$ is obtained.

2.3. Valanis viscoplastic model

Most of plasticity and viscoplasticity theories are derived from the concept of overstress function that requires the existence of yield stress. In some polymers and metals at high temperatures, inelastic deformation occurs at a very low stress level. In such cases, it is quite difficult to accurately define the yield point. Valanis (1971) proposed an endochronic viscoplastic model based on the irreversible thermodynamics concept without a yield surface for isotropic materials. Valanis's viscoplastic model is expressed as:

$$S_{ij}^t = 2G \int_0^z \rho(z - z') \frac{de_{ij}^{vp}}{dz'} dz' \tag{22}$$

where S_{ij}^t and $e_{ij}^{vp,t}$ are the deviators of the stress and viscoplastic strain, respectively, and G is the elastic shear modulus. The kernel $\rho(z)$ is a non-dimensional material function and z appearing in the equation is the intrinsic time. The kernel $\rho(z)$ can be written as:

$$\rho(z) = \rho_0 \delta(z) + \rho_1(z) \tag{23}$$
$$S_y^0 = 2G\rho_0 \tag{24}$$

where ρ_0 is a material constant related to an isotropic hardening, $\delta(z)$ is a Dirac delta function, ρ_1 is a non-singular function that describes a kinematic hardening behavior, and the variable S_y^0 is a material constant which can be related to a hardening parameter. Substituting Eqs. (23) and (24) into Eq. (22) yields:

$$S_{ij}^t = S_y^0 \frac{de_{ij}^{vp,t}}{dz} + 2G \int_0^z \rho_1(z - z') \frac{de_{ij}^{vp,t}}{dz'} dz' \tag{25}$$

The variable z in Eq. (25) is the intrinsic time which is given by

$$dz = \frac{d\zeta}{f(\zeta)} \tag{26}$$

For an isotropic hardening model, the second term of Eq. (25) can be retrieved at all times and Eq. (25) can be rewritten as:

$$S_{ij}^t = S_y^0 \frac{de_{ij}^{vp,t}}{dz} = S_y^0 \frac{de_{ij}^{vp,t}}{d\zeta} f(\zeta) \tag{27}$$

The parameter ζ is an intrinsic time scale that depends on the magnitude of the incremental viscoplastic strain, and is defined as (Valanis, 1971; Khan and Huang, 1995):

$$d\zeta = \sqrt{de_{ij}^{vp,t} de_{ij}^{vp,t}} \tag{28}$$

Using Eq. (27), the yield criterion is defined as:

$$S_{ij}^t S_{ij}^t = S_y^0 \frac{de_{ij}^{vp,t}}{d\zeta} f(\zeta) S_y^0 \frac{de_{ij}^{vp,t}}{d\zeta} f(\zeta) = \left(S_y^0\right)^2 \frac{de_{ij}^{vp,t} de_{ij}^{vp,t}}{(d\zeta)^2} (f(\zeta))^2 = \left(S_y^0\right)^2 (f(\zeta))^2 \tag{29}$$

Material responses during the deformation are categorized as:

$$\begin{aligned} &S_{ij}^t S_{ij}^t \leqslant \left(S_y^0\right)^2 f^2(\zeta) \quad \text{(elastic or viscoelastic response)} \\ &S_{ij}^t S_{ij}^t = \left(S_y^0\right)^2 f^2(\zeta) \quad \text{and} \quad S_{ij}^t de_{ij}^{vp,t} > 0 \quad \text{(viscoplastic response)} \end{aligned} \tag{30}$$

The parameter $f(\zeta)$ is a non-negative function of the intrinsic time scale, ζ, with $f(0) = 0$. If $f(\zeta)$ is equal to 1, Eq. (29) corresponds to the von Mises yield function. The parameter ζ is an intrinsic time scale that depends on the magnitude of the incremental viscoplastic strain (Valanis, 1971; Khan and Huang, 1995), which at the current time t remains as unknown. It is possible to rewrite the intrinsic time scale in terms of known variables, such as current stress, strain, and time. This expression will be discussed later in this manuscript. The intrinsic time scale function $f(\zeta)$ can take various forms (Valanis, 1971), such as:

$$\begin{aligned} &f(\zeta) = 1 + \beta\zeta \quad \text{(Linear form)} \\ &f(\zeta) = a + (1-a)e^{-b\zeta} \quad \text{(Exponential form)} \end{aligned} \tag{31}$$

The parameters β, a, and b are material constants that have to be characterized from experiments. The time derivative of the deviatoric and volumetric strains from Eq. (27), are:

$$\begin{aligned} &de_{ij}^{vp,t} = \tfrac{1}{S_y^o f(\zeta)} S_{ij}^t d\zeta \\ &d\varepsilon_{kk}^{vp,t} = 0 \end{aligned} \tag{32}$$

The incremental total strain is defined by adding the incremental (visco) elastic and viscoplastic components. Using the expression of total incremental strain in Eq. (1), the incremental stress for isotropic material model can be expressed by:

$$dS_{ij}^t = 2G(de_{ij}^t - de_{ij}^{vp,t}) = 2Gde_{ij}^t - \frac{2G}{S_y^o f(\zeta)} S_{ij}^t d\zeta \tag{33}$$

The incremental deviatoric stress is related to the incremental deviatoric visco (elastic) strain in Eq. (32) by the value of G that is time dependent. The time-dependent effect is due to the viscoelastic part. If instead the elastic part is considered, G value is the linear elastic shear modulus. By differentiating equation (29), the endochronic consistency condition is given as:

$$S_{ij}^t dS_{ij}^t = (S_y^o)^2 f(\zeta) f'(\zeta)\, d\zeta \tag{34}$$

Substituting Eq. (33) into Eq. (34) gives:

$$S_{ij}^t\left(2Gd\varepsilon_{ij}^t - \frac{2G}{S_y^o f(\zeta)} S_{ij}^t d\zeta\right) = \left(S_y^o\right)^2 f(\zeta) f'(\zeta)\, d\zeta \tag{35}$$

The strain increases monotonically by setting $f(\zeta)$ to be a monotonic decreasing function of ζ. In Valanis's model, the actual time variable is not present in the formulation, but it is implicitly incorporated in the stress and deviatoric viscoplastic strain at the current time. Thus, the associated time is denoted by the intrinsic time scale ζ. Fig. 1 illustrates schematic representations of uniaxial (1-dimensional) responses of Valanis's viscoplastic model with two different intrinsic time scale functions. It is noted that ζ is the magnitude of the viscoplastic strain at the current time. When a linear intrinsic time scale function is chosen, the rate of the viscoplastic strain decreases as time progresses, approaching a zero value at a longer period (like in creep of viscoelastic solid) (Fig. 1a). When an exponential form is used, depending on the parameter a, we can simulate viscoplastic responses with a decreasing rate, constant rate (like in a steady flow), or rapidly increasing rate (like in a tertiary creep behavior).

The incremental total strain is defined by additively combining the incremental viscoelastic and viscoplastic components. The expression of $d\zeta$ from Eq. (35) is given as:

$$d\zeta = \frac{2G}{S_y^o f(\zeta)\left[2G + S_y^o f'(\zeta)\right]}\left(S_{mn}^t de_{mn}^t\right) \tag{36}$$

The expression $d\zeta$ is written in terms of total incremental deviatoric stress, incremental deviatoric strain, and intrinsic time. This expression is suitable for implementation in a displacement based FE where the components of an incremental total stain are the known variables. An incremental formulation of the Valanis viscoplastic constitutive model is formed by assuming $de \approx \frac{\Delta e}{\Delta t}$. Substituting Eq. (36) to (27) and using the above approximation, the incremental viscoplastic strain is:

$$\begin{aligned} &\Delta e_{ij}^{vp,t} = \tfrac{2G}{\left(S_y^o\right)^2 f^2(\zeta)\left[2G + S_y^o f'(\zeta)\right]}\left(S_{mn}^t \Delta e_{mn}^t\right) S_{ij}^t \\ &\Delta\varepsilon_{kk}^{vp,t} = 0 \end{aligned} \tag{37}$$

The incremental form of the magnitude of the incremental viscoplastic strain of the Valanis model is taken as:

$$\Delta\lambda^t = \frac{2G}{\left(S_y^o\right)^2 f^2(\zeta)\left[2G + S_y^o f'(\zeta)\right]}\left(S_{mn}^t \Delta e_{mn}^t\right)\sqrt{S_{ij}^t S_{ij}^t} \quad i = 1,2,3 \tag{38}$$

It should be noted that the Valanis constitutive model is formulated in terms of stress histories, in which the components of stress are taken as independent variables. In this study, the Valanis model will be integrated to micromechanical models of composites, which will be implemented in a general displacement based FE framework. In such cases, components of strains are the known (independent) variables. We need to determine the corresponding stresses. As the total strains consist of the recoverable elastic (viscoelastic) and irrecoverable viscoplastic components, it is important to simultaneously determine these components and total stresses. Linearized (trial) strains components are obtained and a residual vector consisting of the incremental total strains is defined. Eq. (37) together with Eq. (8) gives the total incremental strain at current time t. It is then necessary to correct residual (error) from the linearized solutions.

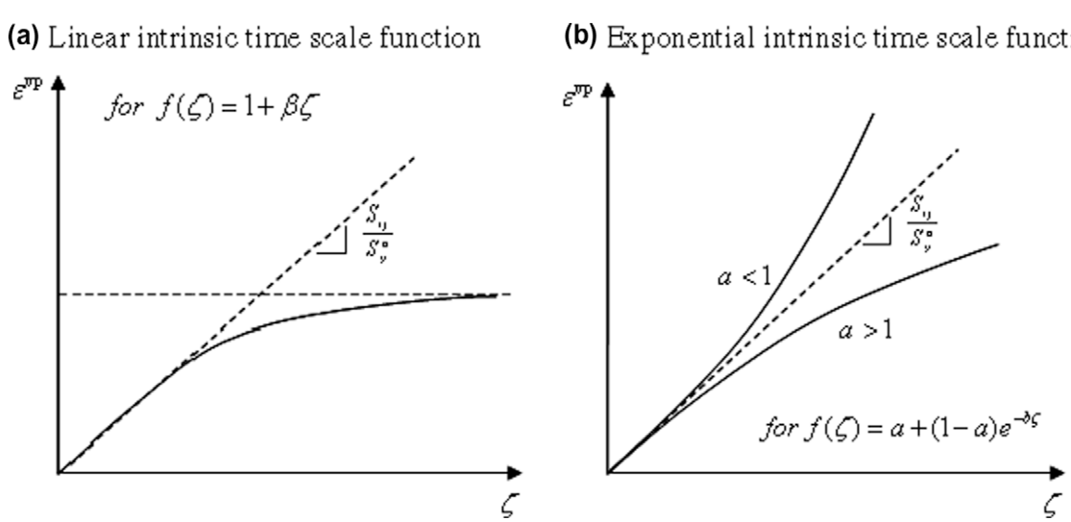

Fig. 1. Responses of Valanis's viscoplastic model.

2.4. Correction algorithm

A time-integration algorithm with linearized predictor and corrector schemes is formulated for the combined nonlinear viscoelastic and viscoplastic constitutive equations. Haj-Ali and Muliana (2004) and Kim and Muliana (2009) have presented predictor–corrector scheme for the Schapery viscoelastic constitutive model and Perzyna viscoplastic model. This study adopts the above predictor–corrector schemes for the combined Schapery–Perzyna and Schapery–Valanis models. At each structural (global) iteration within an incremental time step, $\Delta t^{(m)}$, the components of the incremental strain tensor $\Delta\varepsilon_{ij}^{t,(m)}$ are obtained from the micromechanical level or FE structural level. The goal is to calculate stresses $\sigma_{ij}^{t,(m)}$ and consistent tangent stiffness $C_{ijkl}^{t,(m)}$ at the current time t. At each time increment is also necessary to determine the components of the viscoelastic and viscoplastic strains. The superscript (m) denotes global iteration counter within the current incremental time step. All history variables ($Hist^{t-\Delta t}$) stored from the previous converged solution at time $(t-\Delta t)$, which are $q_{ij,n}^{t-\Delta t},\ q_{kk,n}^{t-\Delta t}$, and $k^{t-\Delta t}$ are passed to the material points and these history variables will be updated once the convergence is achieved. The history variable $k^{t-\Delta t}$ is needed only for the Perzyna's model. For simplicity, the superscript (m) will be dropped. The consistent tangent stiffness matrix C_{ijkl}^{t} at the current time t will be used to provide incremental trial strains for the next time step $(t+\Delta t)$. If at every time increment (Δt) the component of the current total stress tensor σ_{ij}^{t} is the known variables, the stress-dependent incremental viscoelastic strains in Eq. (8) and the incremental viscoplastic strains of Perzyna in Eq. (20) or Valanis in Eq. (37) can be directly calculated. However, the total incremental strains $\Delta\varepsilon_{ij}^{t}$ are the known variables and the total stresses σ_{ij}^{t} are being calculated, while at the same time the strain formulation depends on the current total stresses. One can solve this problem using a linearized stress–strain relation with a constant stress within the time interval (Zienkiewicz and Cormeau, 1972, 1974). The constant stress state can be obtained from the previous converged solution at time $(t-\Delta t)$ and the material parameters can be defined in terms of $\sigma_{ij}^{t-\Delta t}$. At this stage, trial incremental stresses can be obtained from $\Delta\sigma_{ij}^{t,tr}=C_{ijkl}^{t-\Delta t}\Delta\varepsilon_{kl}^{t}$ and total trial stresses are defined by $\sigma_{ij}^{t,tr}=\sigma_{ij}^{t-\Delta t}+\Delta\sigma_{ij}^{t,tr}$. The approximated or trial incremental stresses from the linearized viscoleastic strains, as discussed in Haj-Ali and Muliana (2004), are expressed by:Haj-Ali and Muliana (2004), are expressed by:

$$\Delta S_{ij}^{t(0)}=\Delta S_{ij}^{t,tr}=\frac{1}{\bar{J}^{t,tr}}\left[\Delta e_{ij}^{t}+\frac{1}{2}g_1^{t,tr}\sum_{n=1}^{N}J_n(\exp[-\lambda_n\Delta\psi^t]-1)q_{ij,n}^{t-\Delta t}\right] \tag{39}$$

$$\Delta\sigma_{kk}^{t(0)}=\Delta\sigma_{kk}^{t,tr}=\frac{1}{\bar{B}^{t,tr}}\left[\Delta\varepsilon_{kk}^{t}+\frac{1}{3}g_1^{t,tr}\sum_{n=1}^{N}B_n(\exp[-\lambda_n\Delta\psi^t]-1)q_{kk,n}^{t-\Delta t}\right] \tag{40}$$

The terms $J^{t,tr}$ and $B^{t,tr}$ have the same form as in Eqs. (9) and (10), respectively, but with the non-linear parameters $(g_0, g_1, g_2$, and $a_\sigma)$ are functions of the effective stress from the previous converged step $(\bar{\sigma}^{t-\Delta t})$. Since the trial incremental viscoplastic strain solution is obtained based on the stress state at the previous converged step, this also leads to $\Delta\lambda^{t(0)}=\Delta\lambda^{t,tr}=0.0$ at the beginning of iteration.

Determining the incremental viscoelastic strain $\Delta\varepsilon_{ij}^{ve,t}$ in Eq. (8) and viscoplastic strains $\Delta\varepsilon_{ij}^{vp,t}$ of Perzyna in Eq. (20) or Valanis in Eq. (37) based on the trial stress, will result in a residual strain $\left(\mathbf{R}_{\Delta\varepsilon}^{t,(0)}\right)$. In addition, the linearized stress will cause error in obtaining the plastic multiplier of Perzyna model or Valanis model. In this study, a correction scheme is formulated to minimize errors from the linearization. The residual strains and plastic multiplier are defined in the following linearized equations:

$$R_{\Delta\varepsilon_{ij}}^{t}=\Delta\varepsilon_{ij}^{ve,t}+\Delta\varepsilon_{ij}^{vp,t}-\Delta\varepsilon_{ij}^{t} \tag{41}$$

$$R_{\Delta\lambda}=\Delta\lambda^t-\frac{\Delta t}{\eta_p}\left\langle\left[\frac{(\sigma^{-t-\Delta t}+\Delta\sigma^{-t})-\sigma_y^o-h(k^{t-\Delta t}+\Delta k^t)}{\sigma_y^o}\right]^n\right\rangle e^{-\frac{h}{\eta_p}t}\quad\text{(for Perzyna)} \tag{42}$$

$$R_{\Delta\lambda}=0\quad\text{(for Valanis)} \tag{43}$$

The goal is to calculate $\Delta\sigma_{ij}^{t}$ and update the total stress $\sigma_{ij}^{t}=\sigma_{ij}^{t-\Delta t}+\Delta\sigma_{ij}^{t}$. The Perzyna model depends on the plastic multiplier $\Delta\lambda^t$, which at current time remains as an unknown variable. To determine the total stress in the combined Schapery–Perzyna model, we minimize each component of the residual tensors, $R_{\Delta\varepsilon\ ij}$ and $R_{\Delta\lambda}$. It is noted that the expressions of incremental viscoplastic strains for the Valanis model depends on unknown stress tensor and other material parameters which are known (constant). Thus, it is not necessary to define the residual $R_{\Delta\lambda}$ for the Valanis model as $\Delta\lambda^t$ can be determined once the correct stress has been obtained. This study uses the Newton–Raphson iterative method. At the $(k+1)$ iteration, $\Delta\sigma_{ij}^{t}$ and $\Delta\lambda^t$ are calculated by:

$$\begin{Bmatrix}\Delta\sigma^t\\ \Delta\lambda^t\end{Bmatrix}^{(k+1)}=\begin{Bmatrix}\Delta\sigma^t\\ \Delta\lambda^t\end{Bmatrix}^{(k)}-\left[\frac{\partial\mathbf{R}^{t,(k)}}{\partial\mathbf{X}}\right]^{-1}\begin{Bmatrix}\mathbf{R}_{\Delta\varepsilon}^{t}\\ \mathbf{R}_{\Delta\lambda}^{t}\end{Bmatrix}^{(k)} \tag{44}$$

$$\frac{\partial\mathbf{R}}{\partial\mathbf{X}}=\begin{bmatrix}\frac{\partial\mathbf{R}_{\Delta\varepsilon}}{\partial\Delta\sigma} & \frac{\partial\mathbf{R}_{\Delta\varepsilon}}{\partial\Delta\lambda}\\ \frac{\partial\mathbf{R}_{\Delta\lambda}}{\partial\Delta\sigma} & \frac{\partial\mathbf{R}_{\Delta\lambda}}{\partial\Delta\lambda}\end{bmatrix}_{(7\times7)} \tag{45}$$

Each component of the Jacobian matrix in Eq. (44) is given as follows:

$$\frac{\partial R_{\Delta\varepsilon_{ij}}}{\partial\Delta\sigma_{kl}}=\frac{\partial\Delta\varepsilon_{ij}^{ve,t}}{\partial\Delta\sigma_{kl}^{t}}+\frac{\partial\Delta\varepsilon_{ij}^{vp,t}}{\partial\Delta\sigma_{kl}^{t}} \tag{46}$$

$\partial\Delta\varepsilon_{ij}^{ve,t}/\partial\Delta\sigma_{kl}^{t}$ is given in Haj-Ali and Muliana (2004). For the Perzyna model, $\partial\Delta\varepsilon_{ij}^{vp,t}/\partial\Delta\sigma_{kl}^{t}$ is given as (Kim and Muliana, 2009):

$$\frac{\partial\Delta\varepsilon_{ij}^{vp,t}}{\partial\Delta\sigma_{kl}^{t}}=\frac{2}{3}\frac{\Delta\lambda^t}{(\bar{\sigma}^t)^2}I'_{ijmn}\left(\bar{\sigma}^t I'_{mnkl}-\frac{3}{2\bar{\sigma}^t}S_{mn}^{t}S_{pq}^{t}I'_{pqkl}\right) \tag{47}$$

where $I'_{ijkl}=\delta_{ik}\delta_{jl}-1/3\delta_{ij}\delta_{kl}$

$$\frac{\partial R_{\Delta\varepsilon_{ij}}}{\partial\Delta\lambda^t}=\frac{3}{2\bar{\sigma}^t}I'_{ijkl}S_{kl}^{t} \tag{48}$$

$$\frac{\partial R_{\Delta\lambda}}{\partial\Delta\sigma_{kl}^{t}}=-\frac{\Delta t}{\eta_p}\left\langle\frac{3}{2\sigma^{-t}}\frac{n}{\sigma_y^o}\varphi^{n-1}\Delta\sigma_{kl}^{t}\right\rangle e^{-\frac{h}{\eta_p}t} \tag{49}$$

$$\frac{\partial R_{\Delta\lambda}}{\partial\Delta\lambda^t}=\frac{\Delta t}{\eta_p}\left\langle n\frac{h}{\sigma_y^o}\varphi^{n-1}\right\rangle e^{-\frac{h}{\eta_p}t} \tag{50}$$

When the hardening parameter h is stress-dependent, Eqs. (20), (47), and (48) should be multiplied by $\left(1+\frac{\partial h}{\partial\sigma^{-t}}k^t\right)$ and the following term $-\frac{\Delta t}{\eta_p}\langle\Phi(F)\rangle\frac{\partial}{\partial\Delta\sigma_{kl}^{t}}\left(e^{-\frac{h}{\eta_p}t}\right)$ should be added to Eq. (49). The second term of Eq. (46) for the Valanis model is expressed as:

$$\frac{\partial\Delta\varepsilon_{ij}^{vp,t}}{\partial\Delta\sigma_{kl}^{t}}=\frac{2G}{\left(S_y^o\right)^2 f^2(\zeta)\left[2G+S_y^o f'(\zeta)\right]}\left[2S_{ij}^{t}\Delta\varepsilon_{mn}^{t}I'_{mnkl}\right] \tag{51}$$

Once the convergence is achieved, the consistent tangent stiffness matrix is calculated:

$$C_{ijkl}^{t}=\left[\frac{\partial R_{\Delta\varepsilon_{ij}}^{t}}{\partial\Delta\sigma_{kl}^{t}}\right]^{-1} \tag{52}$$

3. Micromechanical formulation

A micromechanical model is presented for modeling time-dependent and inelastic responses of composites having solid spherical reinforcements. The solid spherical particles are made of linear elastic materials and are assumed to have the same size throughout the composites. The time-dependent and inelastic constitutive model is attributed to the matrix constituent. Fig. 2 illustrates a composite microstructure having randomly distributed solid spherical particles in the homogeneous matrix. The composite microstructure is idealized with periodically distributed arrays of cubic particles, which was previously studied by Muliana and Kim (2007). The geometry representation of the proposed micromechanical model is similar to the one of Method of Cells (MOC), Aboudi et al., 1999. However, the micromechanical formulation does not follow the MOC, which is defined in terms of higher order field variables in each subcell, but is expressed in terms of the average incremental field variables (linearized relation) in the subcells. It is assumed that each particle is fully surrounded by polymeric matrix and direct contact between particles is avoided. A composite representative volume element (RVE) is defined by a cubic particle embedded in the center of the matrix phase of a cubic domain. A one eighth unit-cell consisting of four particle and matrix sub-cells is modeled due to the three-plane symmetry of the RVE. The first sub-cell represents a particle constituent, while sub-cells 2, 3, and 4 represent the matrix constituents. The homogenization scheme is derived in terms of the average strains and stresses in the sub-cells. Perfect bond is assumed at the sub-cell's interface which requires imposing the traction continuity and displacement compatibility at the sub-cells' interfaces. The micromechanical relations provide equivalent homogeneous material responses of the heterogeneous composite and simultaneously predict non-linear behaviors of the individual constituents due to prescribed boundary conditions at the composite (macro) structures. The average stresses and strains in the unit-cell model are defined by:

$$\bar{\sigma}_{ij}^{t} = \frac{1}{V}\sum_{\alpha=1}^{N}\int_{V^{(\alpha)}}\sigma_{ij}^{(\alpha),t}(x_k^{(\alpha)})\,dV^{(\alpha)} \approx \frac{1}{V}\sum_{\alpha=1}^{N}V^{(\alpha)}\sigma_{ij}^{(\alpha),t} \tag{53}$$

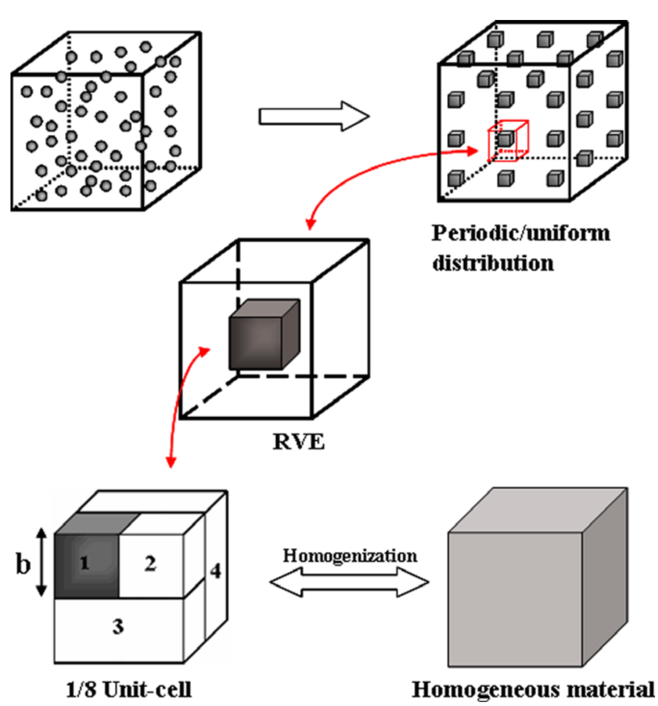

Fig. 2. Representative unit-cell model for the particulate reinforced polymers.

$$\bar{\varepsilon}_{ij}^{t} = \frac{1}{V}\sum_{\alpha=1}^{N}\int_{V^{(\alpha)}}\varepsilon_{ij}^{(\alpha),t}(x_k^{(\alpha)})\,dV^{(\alpha)} \approx \frac{1}{V}\sum_{\alpha=1}^{N}V^{(\alpha)}\varepsilon_{ij}^{(\alpha),t} \tag{54}$$

The superscript (α) denotes the sub-cell number and N is the total number of sub-cells. The stress $\sigma_{ij}^{(\alpha),t}$ and strain $\varepsilon_{ij}^{(\alpha),t}$ are the average stress and strain at current time within each sub-cell. The unit-cell total volume V is given as $V=\sum_{\alpha=1}^{N}V^{(\alpha)}$. To relate the stresses and strains in each sub-cell to the effective stress and strain of the composites, concentration matrices are formulated. The concentration matrices were proposed by Hill (1965) for linear elastic composites. In this study, the micromechanical model is designed to be compatible with displacement based FE structural analyses, in which the effective strains $\bar{\varepsilon}_{ij}^{t}$ are the independent known variables at time t. Due to the non-linear and time-dependent responses in the constituents, the solutions for the deformation fields are performed incrementally. The incremental forms of the effective stress- and strain tensors at the current time are $\bar{\sigma}_{ij}^{t}=\bar{\sigma}_{ij}^{t-\Delta t}+\Delta\bar{\sigma}_{ij}^{t}$ and $\bar{\varepsilon}_{ij}^{t}=\bar{\varepsilon}_{ij}^{t-\Delta t}+\Delta\bar{\varepsilon}_{ij}^{t}$, respectively. The incremental forms are also used for the stress- and strain in each subcell. The sub-cell strain-interaction (concentration) matrix ($\mathbf{B}^{(\alpha),\mathbf{t}}$), which relates the sub-cell average incremental strains at time t, $\Delta\boldsymbol{\varepsilon}^{(\alpha),t}$, to the effective unit-cell average strain at time t, $\Delta\bar{\boldsymbol{\varepsilon}}^{t}$, is defined as:

$$\Delta\varepsilon_{ij}^{(\alpha),t} = B_{ijkl}^{(\alpha),t}\Delta\bar{\varepsilon}_{kl}^{t} \tag{55}$$

Substituting Eq. (55) to (54) gives:

$$\Delta\bar{\varepsilon}_{ij}^{t} = \frac{1}{V}\sum_{\alpha=1}^{N}V^{(\alpha)}B_{ijkl}^{(\alpha),t}\Delta\bar{\varepsilon}_{kl}^{t} \tag{56}$$

It is also seen from Eq. (56) that the $\mathbf{B}^{(\alpha),\mathbf{t}}$ matrices should satisfy the following constraint:

$$\frac{1}{V}\sum_{\alpha=1}^{N}V^{(\alpha)}B_{ijkl}^{(\alpha),t} = \delta_{ik}\delta_{jl} \tag{57}$$

In order to derive the strain-interaction matrices for all sub-cells, the micromechanical relations together with sub-cells constitutive material models must be imposed. Using the strains defined in Eq. (55) and linearized constitutive relations for the constituents, the sub-cell's average stress is:

$$\Delta\sigma_{ij}^{(\alpha),t} = C_{ijkl}^{(\alpha),t}\varepsilon_{kl}^{(\alpha),t} = C_{ijkl}^{(\alpha),t}B_{klrs}^{(\alpha),t}\Delta\bar{\varepsilon}_{rs}^{t} \tag{58}$$

where $\mathbf{C}^{(\alpha),t}$ is the consistent tangent stiffness matrix of the sub-cell at time t which is obtained from the Eq. (52). Substituting Eq. (52) into Eq. (53), the effective stress is given as:

$$\Delta\bar{\sigma}_{ij}^{t} = \frac{1}{V}\sum_{\alpha=1}^{N}V^{(\alpha)}C_{ijkl}^{(\alpha),t}B_{klrs}^{(\alpha),t}\Delta\bar{\varepsilon}_{rs}^{t} \tag{59}$$

The unit-cell effective tangent stiffness matrix $\bar{\mathbf{C}}^{t}$ at time t is determined by:

$$\bar{C}_{ijrs}^{t} = \frac{1}{V}\sum_{\alpha=1}^{N}V^{(\alpha)}C_{ijkl}^{(\alpha),t}\mathrm{B}_{klrs}^{(\alpha),t} \tag{60}$$

The linearized micromechanical relations are satisfied only when all sub-cells exhibit linear elastic responses. Due to the non-linear and time-dependent response in the matrix sub-cells, the linearized micromechanical relations will usually violate the constitutive equations or the non-linear constitutive relations will violate the traction and displacement continuity conditions. The iterative corrector scheme is formulated to minimize errors from the linearization such that both the micromechanical constraints and the non-linear time dependent constitutive equations are satisfied. The goal is to achieve global (structural), micromechanical (material), and constituent (viscoelastic–viscoplastic matrix) con-

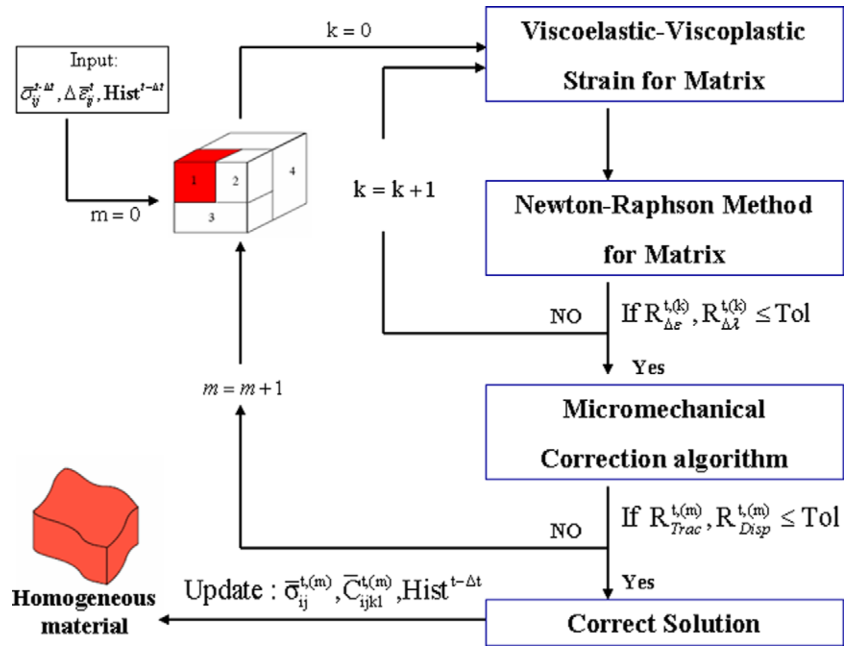

Fig. 3. Summary of homogenization of particle reinforced composite for viscoelastic–viscoplastic responses (k = iteration counter at the sub-cell level, m = iteration counter at the micromechanical model).

vergence simultaneously. Thus, an efficient and accurate numerical algorithm for solving the constitutive material model becomes necessary. The iterative correction scheme for the micromechanical formulation is summarized in Fig 3. Input to this algorithm is the effective incremental strain, previous converged effective stress and history variables. The output is current effective stress, consistent tangent stiffness matrix and updated history variables.

4. Numerical implementation and verification

We present multi-level verification of the viscoelastic–viscoplastic responses of the isotropic constituents and micromechanical model. The time-integration algorithm of the combined Schapery–Perzyna and Schapery–Valanis models are verified using experimental data on high-density polyethylene (HDPE) reported by Lai and Bakker (1995). Available analytical and experimental works in the literature on elastic and viscoplastic responses of solid spherical particle reinforced composites are used for comparisons. FE meshes of unit-cell models of the particle composites that represent microstructures of heterogeneous composites are also generated and the responses obtained from the micromechanical model are compared to the ones obtained using the heterogeneous composites.

4.1. Verification of the time-dependent and inelastic constitutive model

Lai and Bakker (1995) used creep-recovery data of HDPE at various stress levels to calibrate the non-linear viscoelastic and viscoplastic material properties. The plastic deformation increases with

Table 1
Prony series coefficients for the HDPE polymer (Kim and Muliana, 2009).

n	λ_n (s^{-1})	$D_n \times 10^{-4}$ (MPa^{-1})
1	1	2.23
2	10^{-1}	2.27
3	10^{-2}	1.95
4	10^{-3}	3.50
5	10^{-4}	5.50
6	10^{-5}	5.50

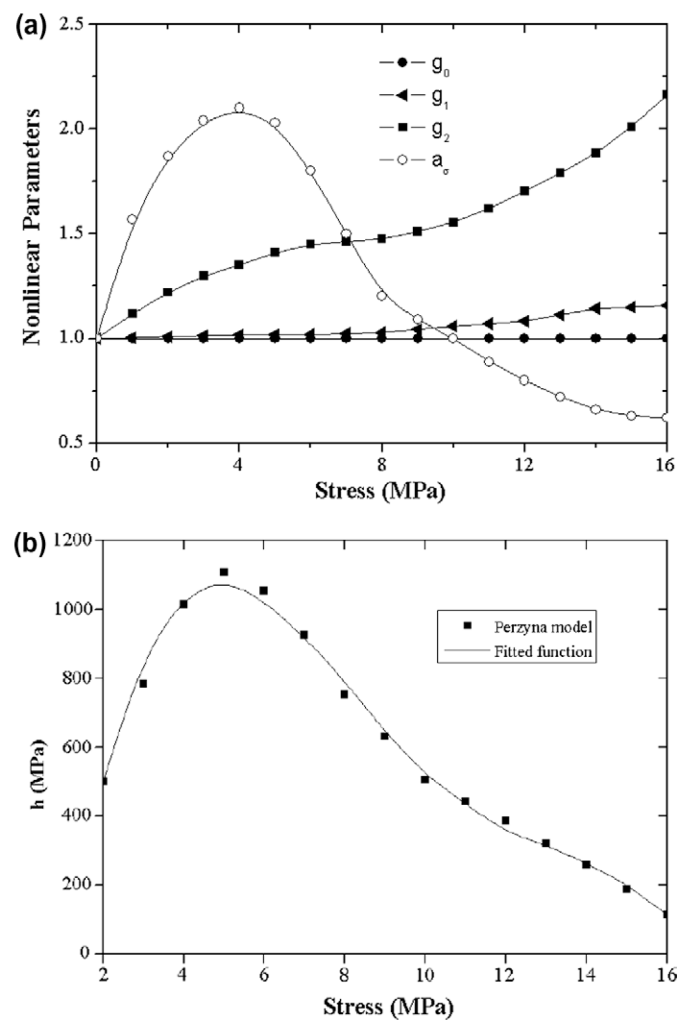

Fig. 4. Non-linear stress dependent parameters: (a) non-linear parameter of the Schapery viscoelastic model (Lai and Bakker, 1995) and (b) hardening parameters (calibrated from the viscoplastic strains, Kim and Muliana, 2009).

time during the creep tests and the increase in the plastic deformation depends on the magnitude of the stress. During the recovery test, as the stress is removed, changes in the strains are due to the viscoelastic part and the plastic deformation, from the last point of the creep test, remains constant. Based on this creep-recovery condition, the viscoplastic strains were separated from the viscoelastic strains. It was shown that for low stresses, the rates of plastic deformation decrease with time and would vanish when a longer creep time is given. For high stress levels the rates of plastic deformation decrease at early time and approach a constant value at later time, which is associated with a steady creep flow. For even higher stresses, the rates of plastic deformation increase rapidly with time. In this study, the calibrated non-linear viscoelastic material parameters by Lai and Bakker (1995) are directly used as input to the numerical algorithm, while the viscoplastic material parameters in the Perzyna or Valanis model are calibrated from the viscoplastic strains during the creep tests at various stresses. The uniaxial elastic modulus and the Poisson's ratio of the tested HDPE are 4535 MPa and 0.3, respectively. The time-dependent material parameters described in Eq. (6) are given in Table 1. The stress-dependent viscoelastic material parameters defined in Eqs. (3) and (4) are given in Fig. 4a. The viscoplastic material constants of Valanis's endochronic model S_y^0 and β are 21 MPa and 0.1, respectively. The parameter S_y^0 describes the increasing rate of the viscoplastic strain and β is time dependent parameter in Eq. (29). The viscoplastic parameter of the Perzyna model η_p and n are 35 MPa s and 1.36, respectively and the hardening parameter is stress dependent. The stress-dependent hardening parameter $h(\sigma)$ of the Perzyna model in Eq. (16) is given in Fig. 4b. The changes in the hardening parameters are associated with the rates of plastic deformation during the creep tests. While the viscosity parameter is taken as constant, to have the rate of plastic deformation increases at later time when the stresses are sufficiently high, it is necessary to have a smaller value for the stress-dependent hardening parameter.

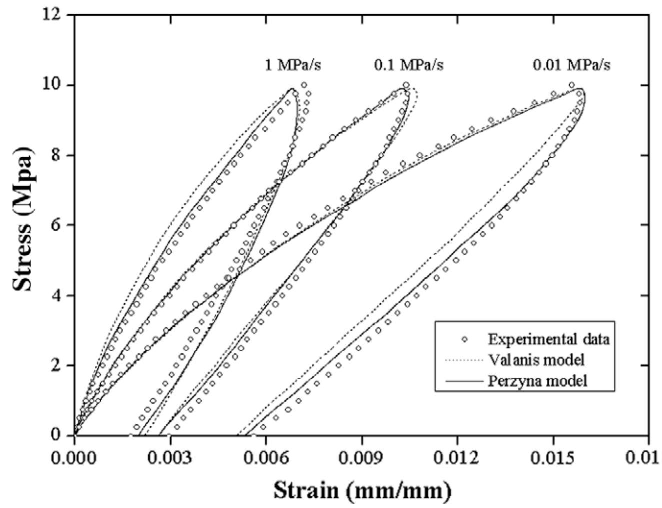

Fig. 6. Stress–strain relations under different constant stress rates (numerical results).

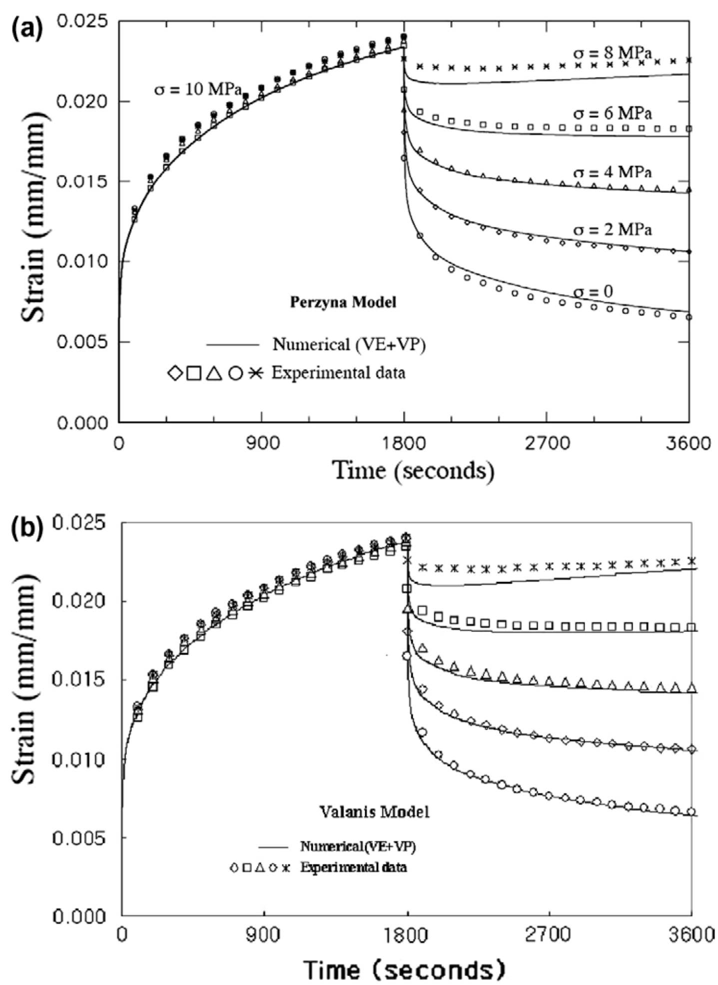

Fig. 5. Total strains from the two-step loading histories: (a) Perzyna model (Kim and Muliana, 2009) and (b) endochronic model.

The numerical algorithm of the Schapery–Valanis models is verified with experimental data from the two-step loading histories, which are given in Fig. 5. This response is also compared to the Schapery–Perzyna model. The first step loading is 10 MPa for 1800 s. This stress is reduced to 8, 6, 4, 2, and 0 MPa, respectively, and is held for another 1800 s. Good comparisons are shown for all cases. Verification of the proposed time-integration algorithm at different loading rates is also performed. Lai and Bakker (1995) conducted tests with constant stress rates for the following uniaxial loading-unloading histories:

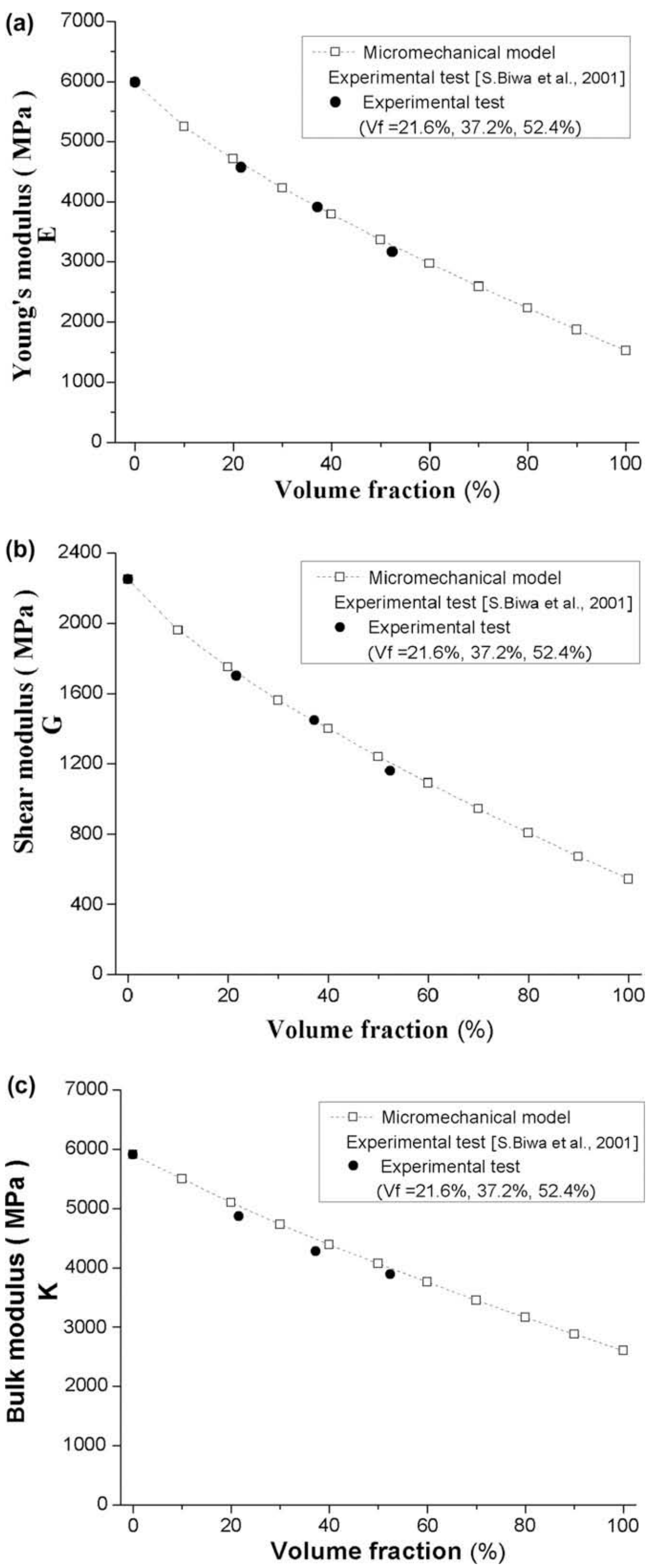

Fig. 7. Effective composite: (a) Young's, (b) shear and (c) bulk moduli with different Vf.

Table 2
Elastic properties of rubber-toughened PMMA composite (Biwa et al., 2001).

Constituents	Bulk moduli K (GPA)	Shear moduli G (GPa)
Rubber particle	2.71	0.56
PMMA (polymethylmethacrylate)	5.91	2.25

Table 3
Elastic properties viscoplastic properties (Pierard et al., 2007).

Constituents	E (GPa)	ν	S_y^o(MPa)	a	b
Particle	400	0.286	7000	0.4	0.00001
Matrix	70	0.33	5000	0.5	0.000015

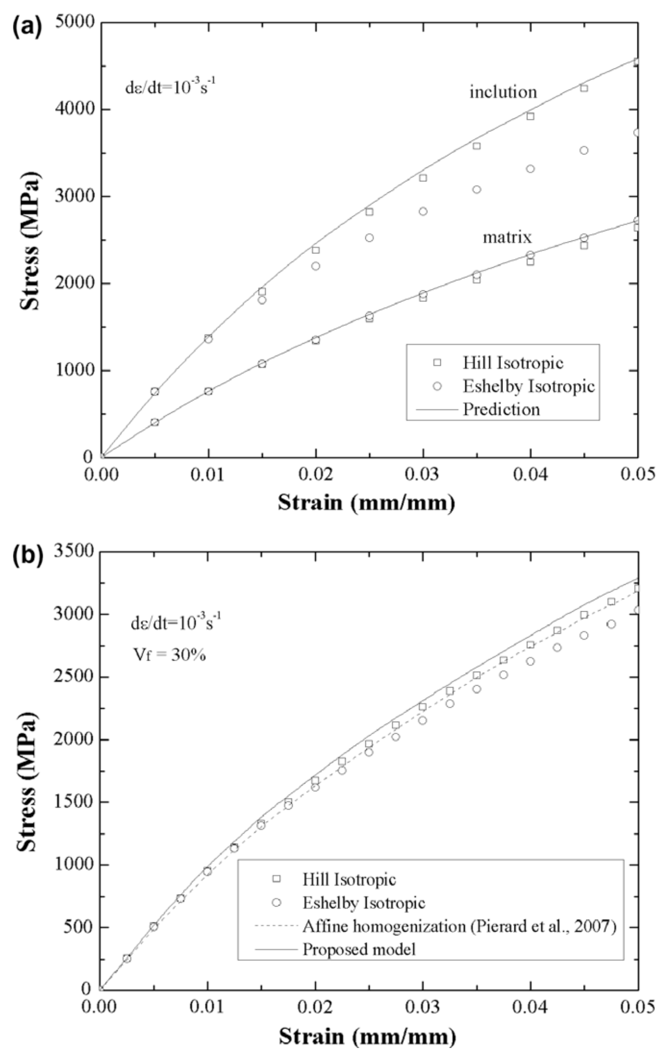

Fig. 8. Stress–strain relation for 10^{-3} strain rate: (a) matrix and inclusion responses and (b) composite response of 30% Vf.

$$\sigma(t) = \begin{cases} rt & 0 \leqslant t \leqslant t_1 \\ r(2t_1 + t) & t_1 \leqslant t \leqslant 2t_1 \end{cases} \tag{61}$$

where r is the constant stress rate and t_1 is the time, when a maximum load $\sigma_{\max}$ is reached, the unloading begins. The stress–strain responses of the Schapery–Perzyna and Schapery–Valanis models are compared with experimental data in Fig. 6. Three different stress rates and $\sigma_{\max}$ = 10 MPa are applied. For the fast loading (1 MPa/s), the viscoelastic–viscoplastic strain is less prominent. For the slower loadings (rates 0.1 MPa/s and 0.01 MPa/s), more pronounced time-dependent and accumulated plastic deformations are observed as expected. For the fast loading of 1 MPa/s, errors about 5% are observed between the Perzyna and Valanis models and experimental data. For the 0.1 MPa/s loading, errors about 2% is observed during loading and errors about 10% is shown for the unloading. For the slowest loading of 0.01 MPa/s, errors about 2% are shown during loading and the unloading part has 5% error for the Perzyna model and 10% error for the Valanis model. Overall the time-integration algorithm for the viscoelastic–viscoplastic response of the Schapery–Perzyna and Schapery–Valanis models predicts the experimental data very well. It is seen that the Perzyna model give better predictions than the Valanis model. The advantage of the Valanis model is it does not required defining an initial yield stress, which is beneficial when plastic deformation occurs at a low stress level. Tolerance of 10^{-6} (1 $\mu\varepsilon$) is chosen for the residual is Eqs. (41)–(43).

Table 4
Elastic properties of glass beads and HDPE.

Constituents	E (MPa)	ν
Glass bead	69,000	0.30
HDPE	4535	0.30

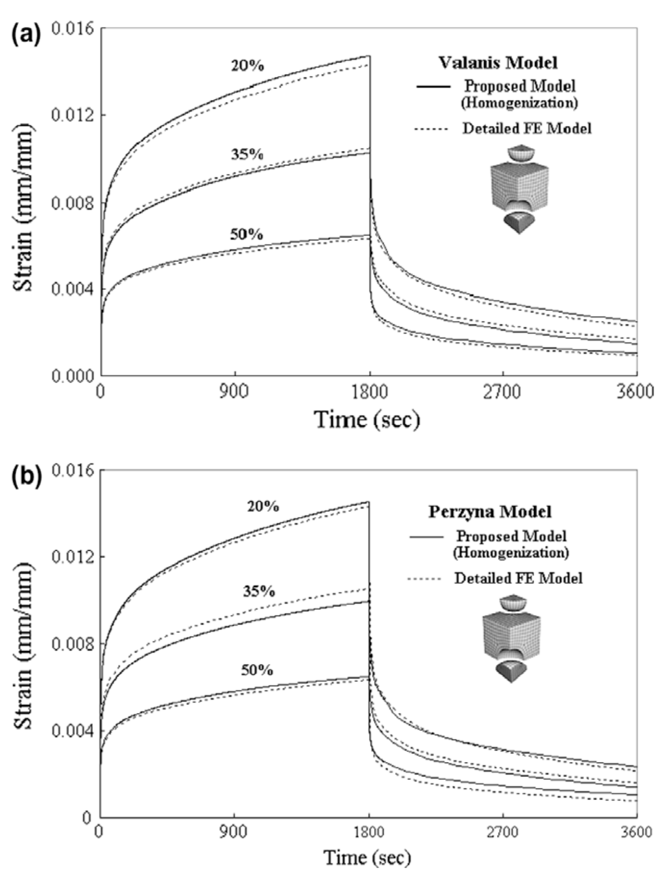

Fig. 9. Creep recovery responses for different volume fraction: (a) Valanis model and (b) Perzyna model.

4.2. Verification of the micromechanical model

The calculated linear elastic moduli of particle composites are first compared with experimental data of Biwa et al. (2001). Fig. 7 shows comparisons of elastic moduli for 21.6%, 37.2%, and 52.4% volume contents of rubber particle-toughened PMMA composite. The constituent properties are obtained from Biwa et al. (2001), which are given in Table 2. The effective Young's, shear, and bulk moduli for several composite volume fractions obtained from the proposed micromechanical model are comparable with experimental data.

To verify the proposed micromechanical model in simulating viscoplastic responses, particle reinforced composites with metal matrix are used. A metal at high temperature exhibits elastic–viscoplastic behaviors. A viscoplastic micromechanical model of metal matrix composites, proposed by Pierard et al. (2007) is used for comparison. Both particles and matrix exhibit viscoplastic responses. The Valanis viscoplastic model is used for the viscoplastic inclusion and matrix. Uniaxial loadings at a strain rate: $10^{-3}\ \mathrm{s}^{-1}$ is applied to composites with 30% particle volume fraction. The elastic properties and Valanis's viscoplastic parameters in Eqs. (31) and

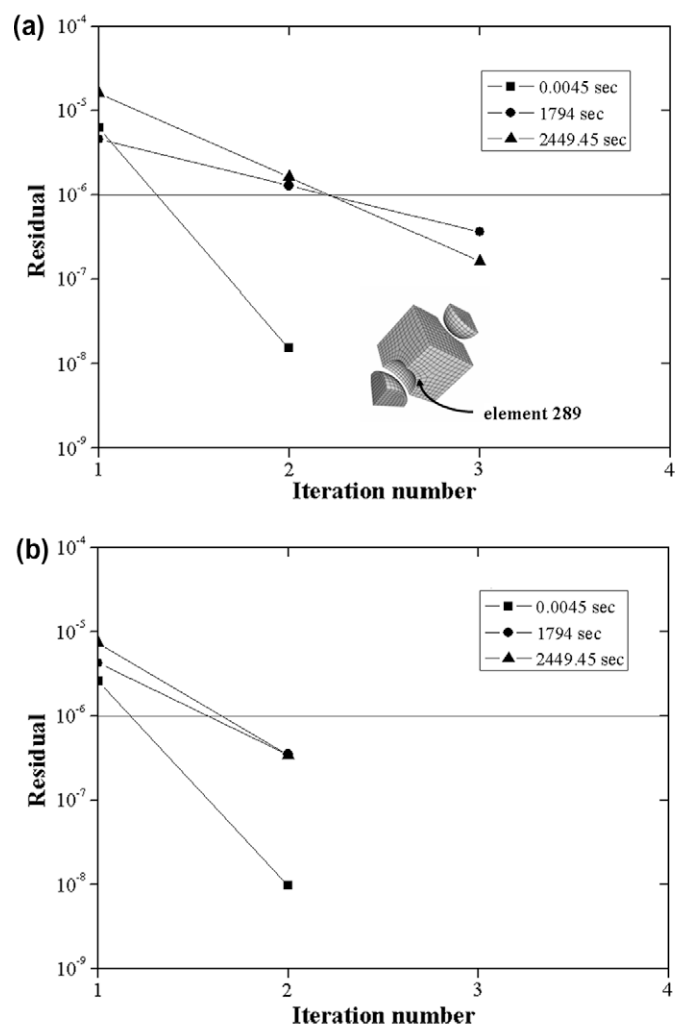

Fig. 10. Convergence behaviors at three times of glass bead/HDPE composite.

(37) are given in Table 3. Fig. 8 presents stress–strain behaviors for each contents and composites subject to $10^{-3}\ s^{-1}$ strain rate. The proposed micromechanical model is comparable to Hill's tensor, and Eshelby's tensor, and affine homogenization of Pierard et al. (2007).

The proposed micromechanical model for the combined visco-elastic–viscoplastic response is examined in terms of its accuracy and efficiency. Experimental data on the combined viscoelastic–viscoplastic behaviors of particle reinforced composites are lacking. Thus, a FE unit-cell model with two-particles that is assumed to closely represent a microstructure of heterogeneous composites is generated to verify the viscoelastic–viscoplastic responses of the proposed micromechanical model. Composites with glass bead particles and HDPE polymer are studied. The glass bead is assumed to be linear elastic and its mechanical properties are given in Table 4. The Schapery–Perzyna and Schapery–Valanis models are considered. The material properties for the HDPE are given in Section 4.1. FE unit-cell models are generated for composites with 20%, 30% and 50% particle contents. Continuum C3D8R elements are used. Total numbers of element and node are 11,737 and 10,206, respectively. Periodic boundary conditions are imposed to the unit-cell model. Creep-recovery loading is simulated by applying a constant stress of 10 MPa for 1000 s and removing the load. The numerical algorithms of the combined Schapery–Valanis model or Schapery–Perzyna model (Section 2) are used for the matrix elements in the unit-cell FE model. Fig. 9 presents creep-recovery responses for composites with different particle contents. The responses obtained from the proposed micromechanical model are comparable to the one of the FE unit-cell models. As expected, adding linear elastic particles reduces creep and plastic strains. Fig. 10 reports the Frobenius norm of the residual vectors in the matrix constituent when the combined Schapery–Perzyna (Eqs. (41) and (42)) is considered during the creep-recovery loading of a composite with 20% particle volume contents. The convergence behaviors at the matrix constituent level are obtained at times 0.0045, 1974, and 2449.345 s. The residuals of the micromechanical model are reported from matrix sub-cell number 2, which has the highest average local effective stress- and initial residual. Residual of the FE unit-cell model is monitored at the matrix element number 289, which is in contact with the particle element and has the highest effective stress- and initial residual, as illustrated in Fig 10. All of these values are monitored during the linearized step at the micromechanical level for the three different times. It is seen that the efficient stress-correction algorithm at the micromechanical model leads to less iteration number at the constituent level, which accelerates the convergence and reduce computational cost.

Large scale structural analysis of particle reinforced composites requires high computing time due to increased number of nodes and elements. In addition, generating FE meshes for composite structures with detailed microstructural characteristics is quite challenging. This study uses the homogenized micromechanical model of particle reinforced composites to analyze effective visco-elastic and viscoplastic responses of a homogenized composite bar, which can be considered as a relatively large-scale structure due to a large number of particles dispersed in homogeneous matrix. Fig. 11 illustrates two FE models that represent homogenized composite and heterogeneous composite microstructures. The heterogeneous composite model incorporates detailed particle geometries. There are 10 spheres, 36 one-quarter spheres, and 8 one-eight spheres. We examine the overall response and local field variables of the homogenized and heterogeneous composites. The longitudinal axis of the bar is along the z-axis. The origin of the coordinate is placed on the left side of the bar. The bar has a length,

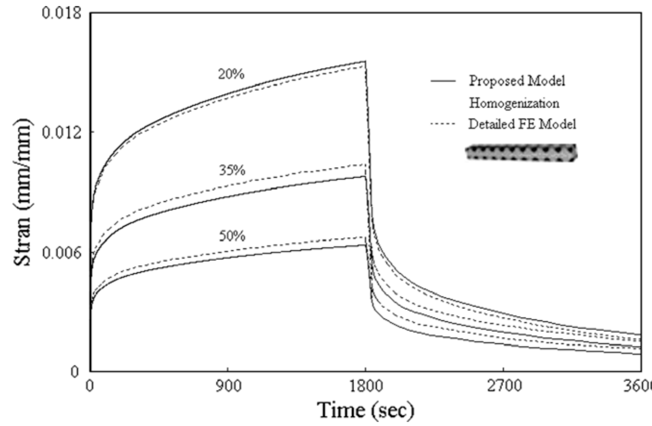

Fig. 12. Creep responses of Valanis model and detailed FE model under 10 MPa.

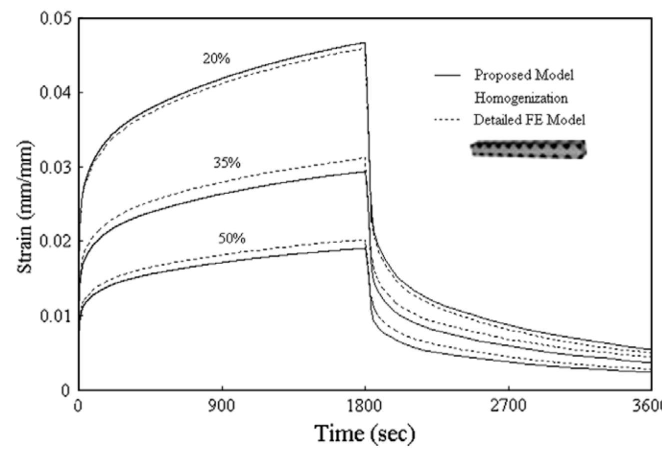

Fig. 13. Creep responses of Valanis model and detailed FE model under 30 MPa.

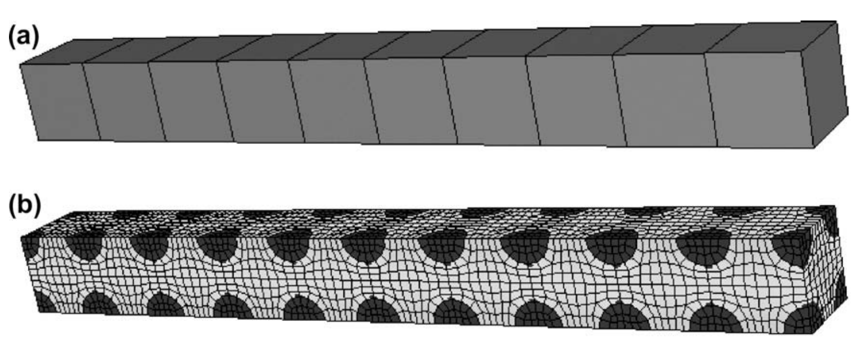

Fig. 11. FE meshes with 20% particle volume contents: (a) homogenized composite bar (#element=10, # node=44) and (b) heterogeneous composite bar (# element=8960, # node=11,201).

Table 5
Comparison of CPU time in the homogenized and heterogeneous composite bars.

	Total elements	Total nodes	CPU time (s)
Homogenized bar (20% Vf)	10	44	65.8
Homogenized bar (35% Vf)	10	44	44.6
Homogenized bar (50% Vf)	10	44	65.3
Heterogeneous bar (20% Vf)	8960	11,201	13,360 (3.71 h)
Heterogeneous bar (35% Vf)	8960	11,201	13,498 (3.75 h)
Heterogeneous bar (50% Vf)	8960	11,201	11,072 (3.06 h)

L = 10 mm and a square cross-section with a side length, S = 1 mm. Composites with different particle contents are subjected to a uniaxial creep-recovery loading. The prescribed boundary conditions are given as:

$$\begin{aligned} &u_x(0,y,z,t)=0 \quad u_y(x,0,z,t)=0 \quad u_z(x,y,0,t)=0 \\ &u_x(S,y,z,t)=\bar{u}_1 \quad u_y(x,S,z,t)=\bar{u}_2 \quad u_z(x,y,L,t)=\bar{u}_3 \\ &\sigma_{zz}(x,y,L,t)=10\ \text{MPa} \quad \text{or } 30\ \text{MPa} \end{aligned} \tag{61}$$

The Schapery–Valanis model is used for the HDPE matrix, while the glass particle is linear elastic. The FE model of the homogenized composite consists of ten C3D8R elements while the heterogeneous composite consists of 8960 elements and 11,201 nodes. Fig. 12 shows responses of the composite bars having 20%, 30%, and 50% particle contents subject to a 10 MPa creep-recovery loading. Fig. 13 shows the creep-recovery responses under 30 MPa. The

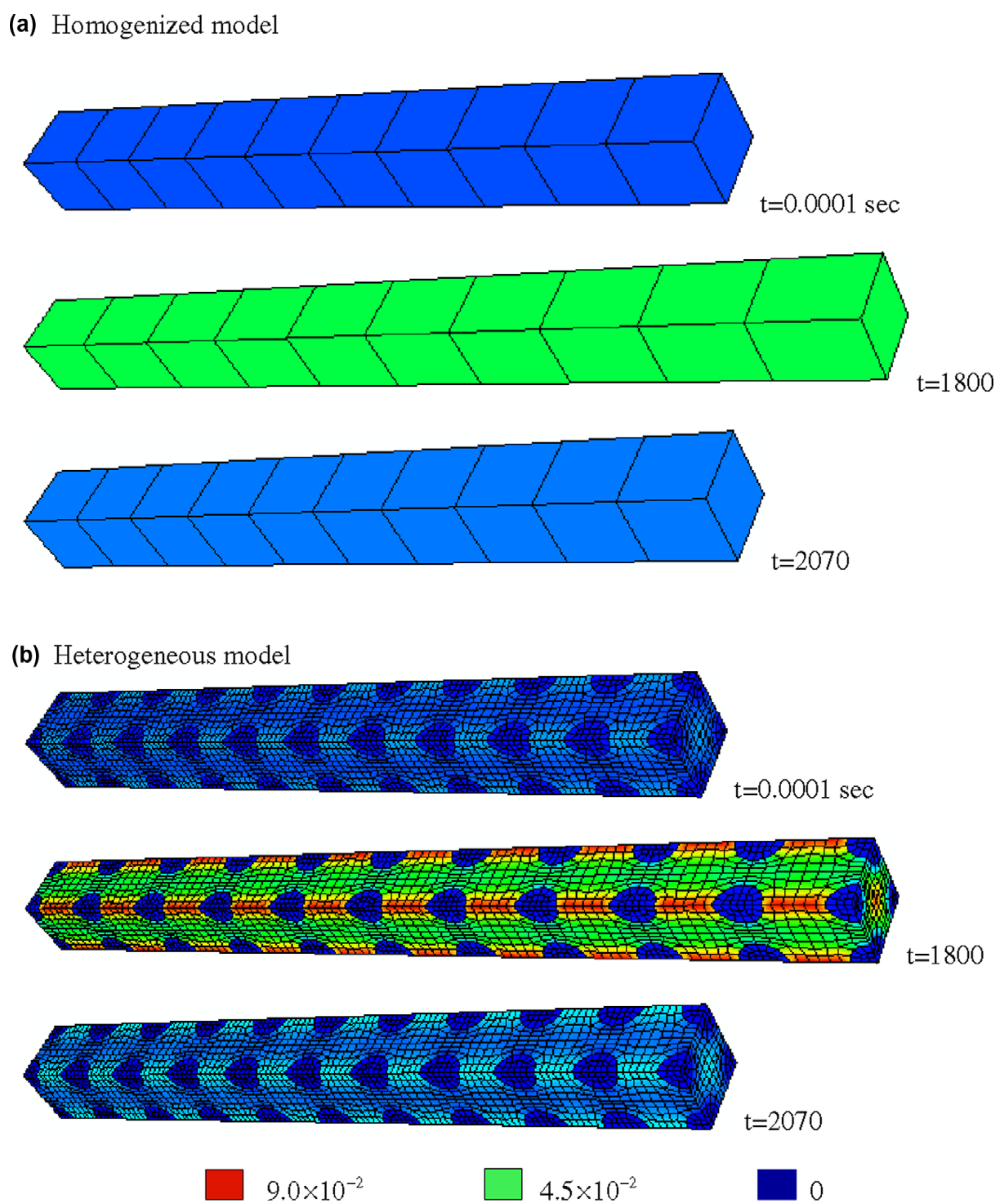

Fig. 14. Maximum principal strain distribution of creep-recovery loading under 30 MPa: (a) proposed model and (b) heterogeneous model.

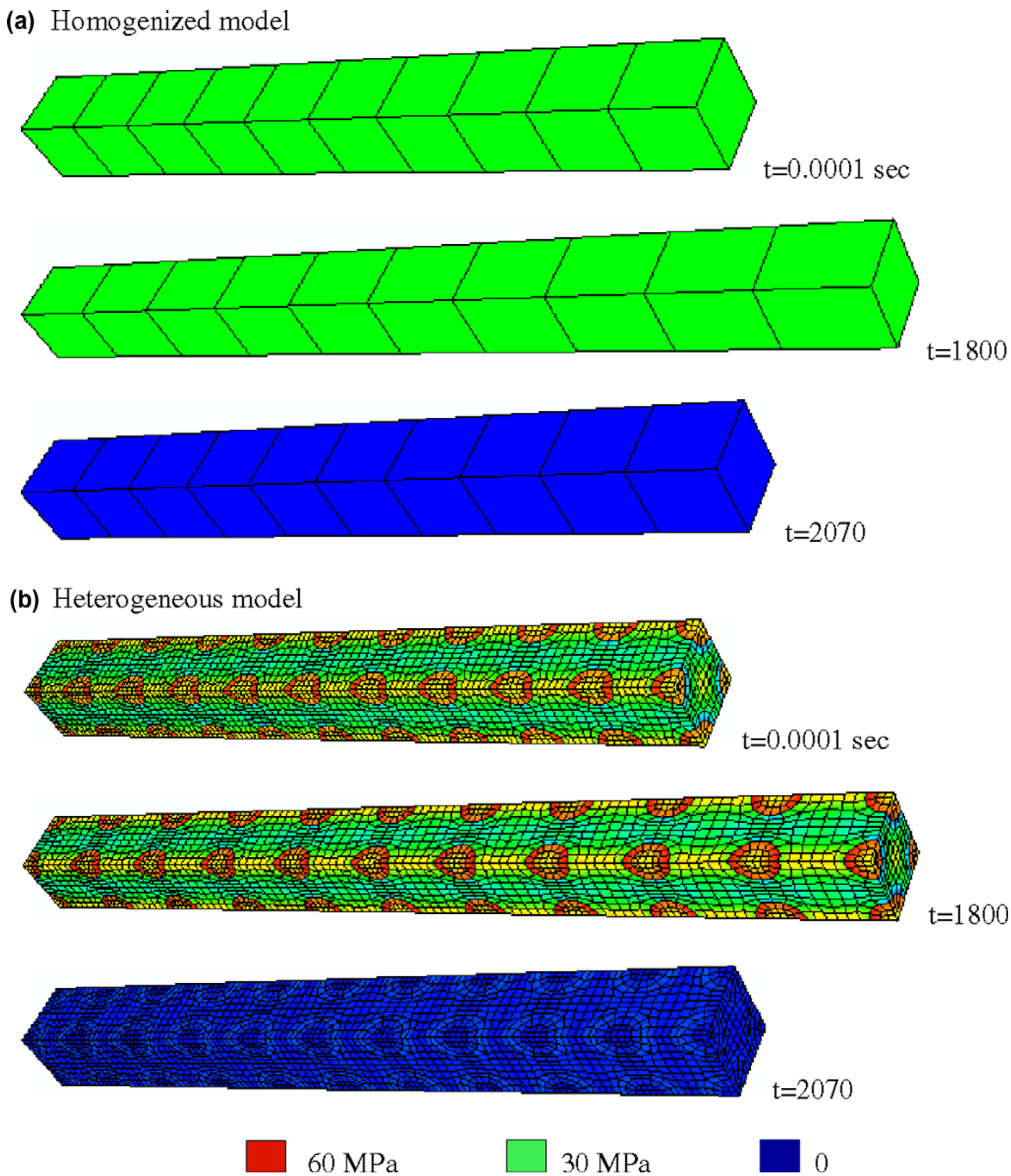

Fig. 15. von-Misses stress distribution of creep-recovery responses under 30 MPa: (a) proposed model and (b) heterogeneous model.

proposed micromechanical model gives good prediction of the overall viscoelastic–viscoplastic responses. Table 5 shows CPU times required for the creep-recovery analyses using the proposed micromechanical model in the homogenized composite bar and heterogeneous FE models of the composite bar. The heterogeneous composites require higher computing time than the homogenized composite bar. Figs. 14 and 15 illustrate local maximum principal strain and von-Misses stress contours of the homogenized and heterogeneous composite bars having 20% particle volume contents at three different times. The scale factor of 10 is used. It is seen that the homogenized composites give prediction of the average (effective) responses but they are limited in characterizing local field variables such as stress concentrations at the particle–matrix inter-phase and at the region between particles. The field variables obtained in the homogenized composites represents average field variables of the matrix and particle constituents.

5. Conclusion

A simplified micromechanical model of particle-reinforced composites has been developed for combined viscoelastic–viscoplastic responses. The viscoelastic–viscoplastic responses are due to the existence of matrix. The proposed micromechanical model provides the capability of predicting the overall time-dependent and inelastic responses of particle reinforced composite and quantifying stress-dependent behavior of the matrix constituent. Multiple time integration algorithms have been developed to solve the time-dependent and inelastic constitutive model in the matrix constituent and to homogenize the effective material responses in the micromechanical model. The micromechanical model gives the average (effective) response but it is limited in capturing local field variables such as stress concentrations at the particle–matrix inter-phase and at the region between particles.

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